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Conditional Expectation and Commutativity in von Neumann Algebras

M. J. MACZYŃSKI

Institute of Mathematics, Technical University, Pl. Jedności Robotniczej 1, 00-661 Warszawa, Poland

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Abstract

Let H be a Hilbert space, $\mathbf{A}$ the von Neumann algebra of all bounded operators on H, $\mathbf{B}$ a von Neumann subalgebra of $\mathbf{A}$, and w a bounded linear functional on $\mathbf{A}$. The functional w is said to commute with $B \in \mathbf{A}$ if $w(AB) = w(BA)$ for all $A \in \mathbf{A}$. It is shown that the map $B \mapsto w(BAB)$ is a complex measure on the orthocomplemented partially ordered set of all orthogonal projections in $\mathbf{B}$ for every $A \in \mathbf{A}$ if and only if w commutes with all members of $\mathbf{B}$. For any $A \in \mathbf{A}$, the conditional expectation of A with respect to $\mathbf{B}$ and w is defined and it is shown that this expectation exists for an Abelian separable $\mathbf{B}$ if w commutes with all members of $\mathbf{B}$. Using Gleason's theorem it is shown that w commutes with $\mathbf{B}$ if and only if the density operator of w commutes with $\mathbf{B}$.

S. Gudder and J. P. Marchand (1972) (in the sequel this paper will be referred to as GM) developed an extensive noncommutative probability theory of von Neumann algebras. Although the basic notion of this theory, conditional expectation with respect to a von Neumann subalgebra, can be defined for arbitrary von Neumann algebras, the condition for the existence of conditional expectation is expressed in terms of the density operator corresponding to the measure in question and requires the use of Gleason's theorem. Consequently, the theory and proofs depend on Gleason's theorem. In this article we would like to show that the definition and condition for the existence of conditional expectation can be expressed without a priori introducing density operators and without referring to Gleason's theorem. To this aim we introduce the notion of a functional commuting with an operator which has interest of its own. Finally, we relate our results to Gleason's theorem.

Let H be a Hilbert space, $\mathbf{A}$ a von Neumann algebra of bounded linear operators on H, $P_{\mathbf{A}}$ the set of all self-adjoint projections in $\mathbf{A}$. A measure on $P_{\mathbf{A}}$ is a non-negative mapping $w: P_{\mathbf{A}} \to R^+$ such that (1) $w(0) = 0$, (2) $w(\Sigma A_i) = \Sigma w(A_i)$ for every finite set of mutually orthogonal projections in $P_{\mathbf{A}}$: If (2) holds for every countable set of mutually orthogonal projections in $P_{\mathbf{A}}$, then

w is said to be a σ-measure. If V is a Banach space, a V-valued measure on $P_{\mathbf{A}}$ is a map $w: P_{\mathbf{A}} \to V$ satisfying (1) and (2). By $\mathbf{A}^*$ we denote the dual space of A (the space of bounded linear functionals on A). $\mathbf{A}$ is assumed to be endowed with the uniform topology, $\mathbf{A}^*$ with the weak-$*$-topology. If $w \in \mathbf{A}^*$, then w restricted to $P_{\mathbf{A}}$ is a complex-valued measure. If w is positive, then w restricted to $P_{\mathbf{A}}$ is a measure. If a positive $w \in \mathbf{A}^*$ restricted to $P_{\mathbf{A}}$ yields a σ-measure, then w is said to be an integral on A.

Definition 1. Let $A \in \mathbf{A}$ and $w \in \mathbf{A}^*$. By Aw we denote the functional in $\mathbf{A}^*$ defined by $Aw(B) = w(AB)$ for all $B \in \mathbf{A}$. Similarly, wA denotes the functional in $\mathbf{A}^*$ defined by $wA(B) = w(BA)$ for all $B \in \mathbf{A}$.

It is clear that for a fixed A the map $B \mapsto w(AB)$ is linear and the functional Aw is bounded, because

$$Aw = \sup_{B \neq 0} \frac{|w(AB)|}{\|B\|} \leqslant \frac{\|w\| \, \|A\| \, \|B\|}{\|B\|} = \|w\| \, \|A\|$$

(similarly for the functional wA).

The maps $(A, w) \mapsto Aw$ and $(w, A) \mapsto wA$ are bilinear maps of $\mathbf{A} \times \mathbf{A}^*$ and $\mathbf{A}^* \times \mathbf{A}$ into A^*, respectively. For simplicity, we shall call these maps left and right multiplications of functionals by operators. It is evident that the operation of multiplication defined above is associative, i.e., $(Aw)B = A(wB)$, $(AB)w = A(Bw)$, and so on. When introducing parentheses, one has to distinguish between the evaluation of a functional in $\mathbf{A}^*$ at some operator in A and the multiplication of that functional by this operator. For example, $wAB = (wA)B$ is a functional in A, whereas $wA(B)$ is a complex number (the value of wA at B). Concluding these remarks let us observe that from the definition of weak-$*$-topology in $\mathbf{A}^*$ it follows that, for a fixed w, the maps $B \mapsto Bw$ and $B \mapsto wB$ are continuous from A into A^*.

Definition 2. Let $w \in \mathbf{A}^*$ and $A \in \mathbf{A}$. We say that w commutes with A if $wA = Aw$. If $X \subset \mathbf{A}$, we say that w commutes with X if $Aw = wA$ for all $A \in X$.

If $w \in \mathbf{A}^*$, from the continuity of our multiplication with respect to both factors it follows that the maps $B \mapsto Bw$ and $B \mapsto wB$ are $\mathbf{A}^*$-valued measures on $P_{\mathbf{A}}$. There arises a natural question under what conditions the map $B \mapsto BwB$ is an $\mathbf{A}^*$-valued measure. The answer is given in the following theorem.

> *Theorem 1.* Let $\mathbf{A}$ be a von Neumann algebra and $\mathbf{B} \subset \mathbf{A}$ a von Neumann subalgebra of $\mathbf{A}$ (containing the identity I). Let w be a functional in $\mathbf{A}^*$. Then the following conditions are equivalent:
> (1) The map $\overline{w}: B \mapsto BwB$ is an $\mathbf{A}^*$-valued measure on $P_{\mathbf{B}}$.
> (2) w commutes with $P_{\mathbf{B}}$.
> (3) w commutes with $\mathbf{B}$.

Proof. It is obvious that (3) implies (2). Assume that (2) holds. Then for any $B \in P_{\mathbf{B}}$ we have $\overline{w}(B) = BwB = (Bw)B = (wB)B = wB^2 = wB$. Taking $B =$

ΣB_i, where B_i is a finite sequence of mutually orthogonal projections in $P_{\mathbf{B}}$, we get

$$w\left(\sum_{i=1}^{n} B_i\right) = w \sum_{i=1}^{n} B_i = \sum_{i=1}^{n} wB_i = \sum_{i=1}^{n} w(B_i)$$

Thus $\bar{w}$ is an $\mathbf{A}^*$-valued measure on $P_{\mathbf{B}}$.

Next assume that (1) holds. Let $B \in P_{\mathbf{B}}$. We have

$$\bar{w}(B + B^{\perp}) = \bar{w}(I) = IwI = w$$

On the other hand, since w is a measure, we have

$$\bar{w}(B + B^{\perp}) = \bar{w}(B) + \bar{w}(B) = BwB + (I - B)w(I - B) = 2BwB - Bw - wB + w$$

Comparing both results we get $2BwB = Bw + wB$. Multiplying on the left and on the right by B and taking into account that $B^2 = B$ we obtain $BwB = wB = Bw$, which shows that w commutes with $P_{\mathbf{B}}$, the set of all projections in $\mathbf{B}$. Hence (2) holds.

It remains to show that (2) implies (3). Using the spectral theorem it is easy to show that w commutes with all self-adjoint elements in $\mathbf{B}$. If $T \in \mathbf{B}$ is arbitrary, T can be represented as $T = A_1 + iA_2$ with $A_1 = \frac{1}{2}(T + T^*)$, $A_2 = -\frac{1}{2}i(T - T^*)$, where A_1 and A_2 are self-adjoint. Now $A_1 w = wA_1$ and $A_2 w = wA_2$ imply $Tw = wT$. Hence (3) holds. This ends the proof of Theorem 1.

For each $B \in P_{\mathbf{B}}$ $\bar{w}(B)$ is a functional in $\mathbf{A}^*$ which can be evaluated at any $A \in \mathbf{A}$. Since in the weak-$*$-topology a sequence of functionals f_n in $\mathbf{A}^*$ is convergent to a functional $f \in \mathbf{A}^*$ if and only if $\lim_{n\to\infty} f_n(A) = f(A)$ for every $A \in \mathbf{A}$, Theorem 1 implies the following corollary.

Corollary 1. The following conditions are equivalent:

(1) The map $B \mapsto BwB(A)$ is a complex-valued measure on $P_{\mathbf{A}}$ for every $A \in \mathbf{A}$.

(2) w commutes with $\mathbf{B}$,

We also have

Corollary 2. If $w \in A^*$ is positive and commutes with $\mathbf{B}$ and A is a positive operator in $\mathbf{A}$, then the map $\bar{w}_A : B \mapsto BwB(A)$ is a measure on $P_{\mathbf{B}}$. If, in addition, w is an integral, then $\bar{w}_A$ is a σ-measure.

Proof. In fact, from Corollary 1 it follows that this map is a complex-valued measure on $P_{\mathbf{B}}$; it remains to show that $\bar{w}_A(B) \geqslant 0$ for all $B \in P_{\mathbf{B}}$. By definition, we have $\bar{w}_A(B) = w(BAB)$. Since $A \geqslant 0$ implies $B^*AB = BAB \geqslant 0$ (see Topping, 1971), and w is positive, we obtain $\bar{w}(B) \geqslant 0$ for all $B \in P_{\mathbf{B}}$. Hence $\bar{w}_A$ is a measure.

To prove the last part of the corollary we apply a theorem of Dixmier (1953) (see also Sakai, 1973, Theorem 1.13.2), which states that for a positive functional w, if w is σ-additive on every countable set of mutually orthogonal projections in a von Neumann algebra $\mathbf{B}$ (i.e., w preserves least upper bounds of countable sets of mutually orthogonal projections in a von Neumann algebra), then w (l.u.b. A_α) = l.u.b. $w(A_\alpha)$ for every uniformly bounded increasing sequence A_α of positive elements in A (these conditions are in fact equivalent). Consequently, if w is an integral commuting with B and $A \in \mathbf{A}$ is positive, then denoting

$$A_n = \sum_{i=1}^{n} B_i$$

we have

$$\overline{w}_A(\Sigma B_i) = \overline{w}_A \text{ (l.u.b. } A_n) = w \text{ (l.u.b. } AA_n) = \text{l.u.b. } w(AA_n)$$

$$= \text{l.u.b. } \sum_{i=1}^{n} w(AB_i) = \sum w(AB_i) = \sum \overline{w}_A(B_i)$$

for every countable set B_i of mutually orthogonal projections in $\mathbf{B}$. This follows from the theorem mentioned above, because the sequence A_n is increasing uniformly bounded and consists of positive elements. Hence w_A is a σ-measure on $P_\mathbf{B}$.

. The corollary above motivates the following definition.

Definition 3. Let $\mathbf{A}$ be a von Neumann algebra, $\mathbf{B} \subset \mathbf{A}$ a von Neumann subalgebra, w a positive functional in $\mathbf{A}^*$ commuting with $\mathbf{B}$. We shall say that two positive operators A_1 and A_2 in A are $(P_\mathbf{B}, w)$-equivalent if the measures $B \mapsto BwB(A_1)$ and $B \mapsto BwB(A_2)$ coincide on $P_\mathbf{B}$.

There arises a question whether for a positive $A \in \mathbf{A}$ there is a positive A_0 that is $(P_\mathbf{B}, w)$-equivalent to A and that belongs to $\mathbf{B}$. If such A_0 exists it is called the conditional expectation of A with respect to $\mathbf{B}$ and w and is denoted by $E_w(A \mid \mathbf{B})$. We refer the reader to GM for a thorough discussion of the properties of this concept and for examples showing that in case $\mathbf{A}$ is an abelian von Neumann algebra arising from a classical probability space, $E_w(A \mid \mathbf{B})$ coincides with the usual conditional expectation of a given $\mathbf{B}$. Here we would like to show that the existence of $E_w(A \mid \mathbf{B})$ depends on the commutativity properties of w with respect to $\mathbf{B}$. Similarly as in GM, we restrict ourselves to considering the case where $P_\mathbf{B}$ is a Boolean algebra of projections (which implies that $\mathbf{B}$ is Abelian).

Before we state the next theorem, let us introduce the following terminology. Let $\mathbf{A} = B(H)$ be the von Neumann algebra of all bounded operators on H. We shall say that $w \in \mathbf{A}^*$ is regular if for every $B \in P_\mathbf{A}$ $w(B) = 0$ implies $BwB = 0$. Later we shall show that Gleason's theorem implies that on a separable Hilbert space every bounded positive functional is regular.

Theorem 2. Let $\mathbf{A} = B(H)$ be the von Neumann algebra of all bounded operators on a Hilbert space H and let $\mathscr{B}$ be a separable Boolean algebra of orthogonal projections in A (generating an abelian von Neumann subalgebra $\mathbf{B} \subset \mathbf{A}$). Let w be a regular integral in $\mathbf{A}^*$ commuting with $\mathscr{B}$. Then for every positive operator $A \in \mathbf{A}$ there is a positive operator A_0 in A all of whose spectral projections belong to $\mathscr{B}$ and which is $(\mathscr{B}, w)$-equivalent to A.

Proof. By Theorem 1 w commutes with $\mathscr{B}$ if and only if w commutes with **B**. Hence from Corollary 2 we infer that $\overline{w}_A : B \mapsto BwB(A) = Bw(A)$ is a σ-measure on $\mathscr{B}$. Since by assumption $\mathscr{B}$ is separable, it is countably generated and consequently there is a σ-homomorphism P from the Borel algebra $B(R)$ on the real line (which is also countably generated and free, see Ramsey, 1966) onto $\mathscr{B}$. This homomorphism is a spectral measure and hence uniquely determines a self-adjoint operator C in A. The composition $w_A \circ P = v_1$ is a σ-measure on $B(R)$ that is absolutely continuous with respect to the measure $v_2 = w^{\mathscr{B}} \circ P$ ($w^{\mathscr{B}}$ denotes the restriction of w to $\mathscr{B}$). In fact, if $v_2(E) = 0$ for some Borel set E, then $w(P(E)) = 0$. Let $P(E) = B$. Since w is regular, $w(B) = 0$ implies $BwB = 0$. Consequently, $\overline{w}_A(B) = BwB(A) = 0$, and $v_1(E) = \overline{w}_A(P(E)) = \overline{w}_A(B) = 0$. Hence $v_1 < v_2$ and the Radon–Nikodym derivative $f(\lambda) = dv_1/dv_2$ exists and is a bounded Borel-measurable function (see Halmos, 1950). Let

$$A_0 = \int f(\lambda) P(d\lambda)$$

Since $f(\lambda) \geqslant 0$, A_0 is positive, and by the definition of P the spectral projections of A belong to $\mathscr{B}$. It remains to show that A_0 is $(\mathscr{B}, w)$-equivalent to A. We have for any $B = P(E)$

$$BwB(A_0) = Bw(A_0) = w(BA_0) = w\left(B \int f(\lambda) P(d\lambda)\right)$$

$$= w\left(P(E) \int f(\lambda) P(d\lambda)\right) = w\left(\int_E f(\lambda) P(d\lambda) = \int_E f(\lambda) w P(d\lambda)\right)$$

$$= \int_E \frac{dv_1}{dv_2}\, dv_2 = v_2(E) = \overline{w}_A(P(E)) = \overline{w}_A(B) = BwB(A).$$

Hence A_0 is $(\mathscr{B}, w)$-equivalent to A. This concludes the proof of Theorem 2.

We shall now give a practical criterion for how to recognize that an integral $w \in \mathbf{A}^*$ commutes with an operator $A \in \mathbf{A}$. This criterion follows from a theorem of Gleason (Gleason, 1957): Let H be a separable Hilbert space with $\dim H > 2$ and let $A = B(H)$. If w is a σ-measure on $P_{\mathbf{A}}$ then there is a unique positive trace class operator W such that $w(A) = \mathrm{Tr}(WA)$ for all $A \in P_{\mathbf{A}}$.

As shown in GM, Gleason's theorem can be extended to obtain the following theorem (Gleason, 1957, Gudder and Marchand, 1972): Let H be a separable Hilbert space with $\dim H > 2$ and let $A = B(H)$. If w is an integral in $\mathbf{A}^*$ then there is a unique positive trace class operator w (called the density operator of w) such that $w(A) = \mathrm{Tr}(WA)$ for all $A \in \mathbf{A}$.

In fact, since an integral $w \in \mathbf{A}^*$ restricted to $P_{\mathbf{A}}$ is a σ-measure, the existence of W follows from Gleason's theorem. It is easy to verify that $w(A) = \mathrm{Tr}(WA)$ for all $A \in P_{\mathbf{A}}$ implies $w(A) = \mathrm{Tr}(WA)$ for all $A \in \mathbf{A}$ (we show this first for self-adjoint elements of $\mathbf{A}$ by way of the spectral theorem similarly as in the proof of Theorem 1, and then for arbitrary $A \in \mathbf{A}$).

With the help of Gleason's theorem we can relate the commutativity properties of w to the commutativity properties of W. Namely, we have the following theorem.

> *Theorem 3.* Let H be a separable Hilbert space with $\dim H > 2$ and let $\mathbf{A} = B(H)$ be the von Neumann algebra of all bounded linear operators on H. Let $w \in \mathbf{A}^*$ be an integral on $\mathbf{A}$ and let W be the density operator of w. Then we have the following:
> (1) w is regular.
> (2) If $WA = AW$ for any $A \in \mathbf{A}$, then $wA = Aw$.
> (3) w commutes with a von Neumann subalgebra $\mathbf{B} \subset \mathbf{A}$ if and only if $W \in \mathbf{B}'$ ($\mathbf{B}'$ denotes the commutant of $\mathbf{B}$).

Proof. To prove (1) we have to show that $w(B) = 0$ for any $B \in P_{\mathbf{A}}$ implies $BwB = 0$. If $w(B) = 0$ then $w(B) = \mathrm{Tr}(WB) = \mathrm{Tr}(WB^2) = \mathrm{Tr}(BWB) = 0$. Since W is positive and B self-adjoint, BWB is also positive and consequently $BWB = 0$. Now for every $A \in \mathbf{A}$, $BwB(A) = w(BAB) = \mathrm{Tr}(WBAB) = \mathrm{Tr}(BWBA) = 0$, i.e., $BwB = 0$. Hence (1) holds.

Next assume that $WA = AW$ for some $A \in \mathbf{A}$. Then for all $B \in \mathbf{A}$ $wA(B) = w(BA) = \mathrm{Tr}(WBA) = \mathrm{Tr}(AWB) = \mathrm{Tr}(WAB) = w(AB) = Aw(B)$, which implies $wA = Aw$. Hence (2) holds.

Property (2) implies that if $W \in \mathbf{B}'$ then $wA = Aw$ for all $A \in \mathbf{B}$, i.e., w commutes with $\mathbf{B}$. Hence (3) holds one way. Conversely, assume that w commutes with $\mathbf{B}$. In particular, this implies that $wB = Bw$ for any projection $B \in P_{\mathbf{B}}$. Multiplying this identity on the left and on the right by B we get $BwB = wB$ $BwB = Bw$. Adding side by side we get $2BwB = wB + Bw$. This implies $2w(BAB) = w(AB) + w(BA)$ for all $A \in \mathbf{A}$, and consequently $2\mathrm{Tr}(WBAB) = \mathrm{Tr}(WAB) + \mathrm{Tr}(WBA)$, or $2\mathrm{Tr}(BWBA) = \mathrm{Tr}((BW + WB)A)$. Taking $A = P_\phi$ the one-dimensional projection on the unit vector ϕ, $\phi \in H$, we infer that $(2BWB\phi, \phi) = ((BW + WB)\phi, \phi)$ for all $\phi \in H$, $\|\phi\| = 1$. Since BWB and $WB + BW$ are self-adjoint, the two quadratic forms coincide on the unit sphere of H, which implies that $2BWB = BW + WB$ for all $B \in P_{\mathbf{B}}$ (Kato, 1966). Multiplying on the left and on the right by B we obtain $BWB = WB = BW$, which shows that W commutes with all projections in $\mathbf{B}$. Reasoning analogously as in the proof of Theorem 1, we show that W commutes with all members of B, i.e., $W \in B'$. Hence (3) holds. This ends the proof of Theorem 3.

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